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PERIODIC AND ALMOST-PERIODIC SOLUTIONS OF DIFFERENTIAL SYSTEMS

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PERIODIC AND ALMOST-PERIODIC SOLUTIONS OF DIFFERENTIAL SYSTEMS /375

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This article discusses some problems concerning the existence of periodic and almost-periodic solutions of certain differential systems.

In this article, we shall consider some problems concerning the existence of periodic and almost-periodic solutions of certain differential systems. We shall use the fixed-point method (in the form of Banach's theorem) and the Lyapunov-function method.

1. Let $A(t)$ denote an $n \times n$ matrix satisfying the following two conditions:

1° $A(t)$ is continuous and ω-periodic;
2° the system

$$\dot{x} = A(t)x, \tag{1.1}$$

where $x \in R^n$, has no ω-periodic solutions except $x(t) \equiv 0$.

Let us denote by $X(t)$ the fundamental matrix of the system (1.1) such that $X(0) = E$. If $x = x(t)$ is a solution of the system (1.1), then $x = x(t + \omega)$ is also a solution.

We know that

$$X(t + \omega) = X(t) \cdot X(\omega).$$

On the other hand, the matrix $B = X(\omega) - E$ is such that $B = \det(X(\omega) - E) \neq 0$.

*Numbers in the margin indicate pagination in the foreign text.

To see this, let us suppose that $\det B = 0$. Then, there exists a vector $h \neq 0$ such that

$$Bh = (X(\omega) - E)h = 0.$$

The function $x(t) = X(t)h$ is a solution of the system (1.1), and it is obvious that $x(t) \not\equiv 0$. But

$$x(t + \omega) = X(t + \omega)h = X(t)X(\omega)h = X(t)h = x(t)$$ /376

which constitutes a contradiction.

Let us also note that

$$B^{-1}X(\omega) = X(\omega)B^{-1}, \quad X(\omega)B = BX(\omega),$$

which follows from the equation

$$B^{-1}X(\omega) = (X^{-1}(\omega) - E)X(\omega) = E - X(\omega) = X(\omega)(X^{-1}(\omega) - E) = \\ = X(\omega)B^{-1}$$

and the equation

$$X(\omega)B = X(\omega)(X(\omega) - E) = X(\omega)(E - X^{-1}(\omega))X(\omega) = \\ = (X(\omega) - E)X(\omega) = BX(\omega).$$

Since

$$X(t + \omega) = X(t)X(\omega)$$

it follows that

$$X^{-1}(t + \omega) = X^{-1}(\omega)X^{-1}(t).$$

By virtue of conditions 1° and 2°, the system

$$\dot{x} = A(t)x + f(t), \tag{1.2}$$

where $f(t)$ is an ω-periodic function, has exactly one ω-periodic solution. More precisely, one can prove the

Lemma. The function

$$u(t) = -X(t)B^{-1}X(\omega)\int_t^{t+\omega} X^{-1}(s)f(s)ds \tag{1.3}$$

is an ω-periodic solution of the system (1.2).

This lemma can be proven by showing that $u(t+\omega) = u(t)$ and that

$$\frac{du}{dt} = A(t)u + f(t).$$

The proof can be found, for example, in [3] and [4].

In what follows, we shall consider the system

$$\dot{x} = A(t)x + \lambda g(t; x), \tag{1.4}$$

where λ is a real parameter and $g(t; \cdot)$ is an operator defined on $C(-\infty, \infty)$, for $t \in (-\infty, \infty)$ (that is, on the space of continuous bounded functions defined on $(-\infty, \infty)$, that have values in R^n). To emphasize that the image, under the operator g, of a function $x(t) \in C(-\infty, \infty)$ is also a function of t, we shall also write $g(t; x) = (Gx)(t)$. /377

Let $P(\omega)$ denote the Banach space of ω-periodic functions defined on the real axis with norm $\|x\|_P = \sup\limits_{t \in (-\infty, \infty)} \|x(t)\|$.

Definition 1.1. The operator $g(t; x)$ defined for $(t, x) \in (-\infty, \infty) \times P(\omega)$ is said to be ω-periodic if $g(t; \varphi) = (G\varphi)(t) \in P(\omega)$, for $\varphi(t) \in P(\omega)$

With the aid of the lemma and the definition, we can easily prove the following theorem, which gives the form of the ω-periodic solution of the system (1.4).

Theorem 1.1. If the matrix $A(t)$ satisfies conditions 1° and 2° and the operator $g(t; x)$ is ω-periodic, then any ω-periodic solution of the system (1.4) is an ω-periodic solution of the system of integral equations

$$y(t) = -\lambda X(t) B^{-1} X(\omega) \int_t^{t+\omega} X^{-1}(s) g(s; y) ds, \tag{1.5}$$

and vice versa.

Proof: Let $x = u(t)$ denote an ω-periodic solution of the system (1.4). Define

$$h(t) = \lambda g(t; u) = \lambda (Gu)(t).$$

Then, $x = u(t)$ is an ω-periodic solution of the system

$$\dot{x} = A(t)x + h(t). \tag{1.6}$$

Since (1.6) has a unique ω-periodic solution, it follows on the basis of the lemma that

$$u(t) = -X(t)B^{-1}X(\omega)\int_t^{t+\omega} X^{-1}(s)h(s)ds,$$

Consequently, $u(t)$ is an ω-periodic solution of the system (1.5).

Now suppose that $x = v(t)$ is an ω-periodic solution of the system (1.5). Then, the function

$$l(t) = \lambda g(t; v) = \lambda(Gv)(t)$$

is ω-periodic. According to the lemma, the function

$$v(t) = -X(t)B^{-1}X(\omega)\int_t^{t+\omega} X^{-1}(s)l(s)ds$$

is the ω-periodic solution of the system

$$\dot{x} = A(t)x + l(t),$$ /378

that is, $x = v(t)$ is the solution of the system (1.4), which proves Theorem 1.1.

Consider again the system of integral equations (1.5). Let us now write

$$(Ty)(t) = -\lambda X(t)B^{-1}X(\omega)\int_t^{t+\omega} X^{-1}(s)g(s; y)ds.$$

We note that, if conditions 1° and 2° are satisfied and the operator $g(t; y)$ is ω-periodic, then the operator T is such that $T(P(\omega)) \subset P(\omega)$.

In what follows, we shall assume that the operator T is defined on the space $P(\omega)$.

Theorem 1.2. If the matrix $A(t)$ satisfies conditions 1° and 2° and the operator $g(t; x)$ is ω-periodic and satisfies the condition

$$\|g(t; x) - g(t; y)\| \leq L\|x - y\|_P,$$

for every pair $x, y \in P(\omega)$, then there exists a unique ω-periodic solution of the system (1.4) for λ sufficiently small.

Proof. We have already pointed out that $T(P(\omega)) \subset P(\omega)$. Since the space $P(\omega)$ is a Banach space, it remains to show that the operator T is a contraction operator, which will enable us to apply Banach's fixed-point theorem, taking $P(\omega)$ as the fundamental space with the metric

$$\rho(u, v) = \sup_{t\in(-\infty, \infty)} \|u(t) - v(t)\| = \|u - v\|_P.$$

Let Δ denote the domain $0 \leqslant t \leqslant \omega,\ 0 \leqslant s \leqslant 2\omega$ and define

$$\sup \|X(t+\omega)B^{-1}X^{-1}(s)\| = m.$$

for t, s $\in \Delta$

If the functions $u(t)$ and $v(t)$ belong to the space $P(\omega)$, we have

$$\|(Tu)(t) - (Tv)(t)\| \leqslant |\lambda| \int_t^{t+\omega} \|X(t)B^{-1}X(\omega)X^{-1}(s)\|\,\|g(s;u) -$$

$$- g(s;v)\|ds \leqslant |\lambda| m \int_0^{\omega} L\|u-v\|_P\,ds = |\lambda| m \int_0^{\omega} L\rho(u,v)\,ds = |\lambda|\, m\omega\rho(u,v),$$

and, consequently,

$$\rho((Tu)(t),\ (Tv)(t)) \leqslant |\lambda| Lm\omega\rho(u,v).$$

If we take

$$|\lambda| < \frac{1}{Lm\omega},$$ /379

It follows that T is a contraction operator and Theorem 1.2 is proven.

Remarks: 1. In the particular case in which

$$g(t;x) = \int_0^{\alpha(t)} k(t,s,x(s))\,ds$$

where $\alpha(t)$ is an ω-periodic function and the vector-values function $k(t,s,x)$ is ω-periodic with respect to t and continuous in the domain

$$\Delta_1 = \{t \in (-\infty,\infty), x \in R^n, |s| < r\},$$

the number r being a bound on $|\alpha(t)|$, similar results have been established by I. V. Bykov and M. Imanaliev [3].

2. Another interesting special case is that in which $g(t;x) = g(t;x_t)$, where $\alpha < s \leqslant t$, and $x_t = x(s)$, that is, $g(t;x)$ is a Volterra operator.

2. In this section, we shall consider the problem of the existence of almost-periodic solutions for a certain system of differential equations by using a particular Lyapunov function and the fixed-point method.

Let us consider first the system of differential equations

$$\dot{x} = f(t, x) + h(t), \tag{2.1}$$

where $f(t, x) \in C((-\infty, \infty) \times R^n)$, with $f(t_0, x) \in C^1(R^n)$ for every point t_0 in $(-\infty, \infty)$, and $h(t) \in C(-\infty, \infty)$. Suppose that both functions assume values in R^n. Let $\Lambda(t, x)$ denote the greatest eigenvalue of the matrix

$$J_s(t, x) = \frac{1}{2}[Af'_x(t, x) + (Af'_x(t, x))^*],$$

where $A = (a_{ij})_{1 \leqslant i, j \leqslant n}$ is a constant symmetric positive-definite matrix.

Theorem 2.1. Suppose that the following conditions are satisfied:

(a) $\Lambda(t, x) \leqslant -\alpha < 0$ for $(t, x) \in (-\infty, \infty) \times R^n$;
(b) $\|f(t, 0)\| \leqslant \beta$ et $\|h(t)\| \leqslant \gamma$.

Then the system (2.1) has a unique bounded solution $x = x(t)$, $(\|x(t)\| \leqslant R)$. If in addition the functions $f(t, x)$ and $h(t)$ are almost-periodic with respect to t uniformly with respect to x for $\|x\| \leqslant R$, then the bounded solution $x = x(t)$ is also almost-periodic.

Proof: We note first of all that the proof of the theorem is similar to that of Theorem 2, due to Demidovich [2] with a necessary modification caused by the term $h(t)$. Define $V(x) = (Ax, x)$. We have

$$\alpha'\|x\|^2 \leqslant (Ax, x) \leqslant \alpha''\|x\|^2,$$ /380

where α' and α'' are positive constants.

If $x = x(t)$ is a solution of the system (2.1), we obtain

$$\dot{V}(x(t)) = 2(A\dot{x}(t), x(t)) = 2(Af(t, x), x) + 2(Ah(t), x) =$$
$$= 2(A[f(t, x) - f(t, 0)], x) + 2(Af(t, 0), x) + 2(Ah(t), x).$$

From Demidovich's lemma [2] and condition (a), we obtain

$$(A[f(t, x) - f(t, 0)], x) \leqslant \Lambda(t, x)\|x\|^2 \leqslant -\alpha\|x(t)\|^2 \leqslant -\frac{\alpha}{\alpha''}V(x(t))$$
$$= -kV(x(t)).$$

From condition (b), we obtain

$$\dot{V}(x(t)) \leqslant -2kV(x(t)) + 2\|A\|(\beta + \gamma)\|x\| \leqslant -2kV(x(t)) +$$
$$+ 2\|A\|\frac{(\beta + \gamma)}{\alpha'}V^{1/2}(x(t))$$

or

$$\frac{1}{2}\dot{V}(x(t)) \leqslant -kV(x(t)) + rV^{1/2}(x(t)), \qquad (2.2)$$

where

$$r = \|A\|\frac{(\beta+\gamma)}{\alpha'}.$$

Using the result on differential inequalities (for example [1], p. 106), we find

$$V(x(t)) \leqslant \left[\sqrt{V(\overline{x}(t_0))}\, e^{-k(t-t_0)} + \frac{r}{k}\right]^2,$$

which implies the existence of bounded solutions.

To prove the uniqueness of the bounded solution $x = x(t)$, let us consider the function $V = (A(x-\overline{x}),\ x-\overline{x})$, where $x = \overline{x}(t)$ is another bounded solution of the system (2.1). Then

$$\frac{1}{2}\dot{V}(t) = (A[f(t,x(t)) - f(t,\overline{x}(t))],\ x(t)-\overline{x}(t)) \leqslant -\|x(t)-\overline{x}(t)\|^2,$$

so that

$$\dot{V}(t) \leqslant -2\frac{\alpha}{\alpha''}V(t),$$

that is,

$$\|x(t)-\overline{x}(t)\| \leqslant \sqrt{\frac{\alpha}{\alpha''}}\,\|x(t_0)-\overline{x}(t_0)\|\, e^{-\alpha/\alpha''(t-t_0)},\quad t \geqslant t_0. \qquad (2.3)$$ /381

By letting t_0 approach $-\infty$, we see that $x(t) = \overline{x}(t)$ for all t in $(-\infty, \infty)$, so that the bounded solution is unique.

Let us now prove the second part of the theorem. We note first that since $f(t, x)$ and $h(t)$ are almost-periodic, condition (b) is satisfied. Suppose that $\|x(t)\| < R$. Since $f(t, x)$ is a uniformly almost-periodic function of t for $\|x\| \leqslant R$, it follows that, for all $\eta > 0$, there exists a $\sigma(\eta) > 0$ such that every real interval of length σ includes at least one number τ such that

$$\|f(t+\tau, x) - f(t, x)\| < \eta^2,\quad -\infty < t < \infty,\ \|x\| \leqslant R$$

and for such τ

$$\|h(t+\tau) - h(t)\| \leqslant \eta^2.$$

Let us set

$$V(t) = (A[x(t+\tau) - x(t)],\ x(t+\tau) - x(t)).$$

Then,

$$\frac{1}{2}\dot{V}(t) = (A[f(t+\tau, x(t+\tau)) + h(t+\tau) -$$
$$- f(t, x(t)) - h(t)],\ x(t+\tau) - x(t)) =$$
$$= (A[f(t+\tau, x(t+\tau)) - f(t+\tau, x(t))],\ x(t+\tau) - x(t)) +$$
$$+ (A[f(t+\tau, x(t)) - f(t, x(t))],\ x(t+\tau) - x(t)) +$$
$$+ (A[h(t+\tau) - h(t)],\ x(t+\tau) - x(t)) \leqslant -\frac{\alpha}{\alpha''}V(t) + 4\|A\|R\eta^2$$

Therefore,

$$V(t) \leqslant V(t_0)\, e^{-2\alpha/\alpha''(t-t_0)} + \frac{4\|A\|\, R\, \alpha''}{\alpha}\eta^2 \quad \text{for} \quad t \geqslant t_0. \tag{2.4}$$

By letting t_0 approach $-\infty$, we obtain

$$V(t) \leqslant \frac{4\|A\|}{\alpha} R\alpha''\eta^2,$$

that is,

$$\|x(t+\tau) - x(t)\| < \lambda\eta, \tag{2.5}$$

where $\lambda = 2\sqrt{\|A\|\, R\alpha''/\alpha\alpha'}$, which proves that the solution $x = x(t)$ is almost-periodic.

Remark: A similar result can be proven in the case in which $f(t, x)$ is a periodic function of t uniformly with respect to x for $\|x\| \leqslant R$ and $h(t)$ is periodic. /382

In what follows, we shall establish the existence of almost-periodic solutions for a perturbed system of the form

$$\dot{x} = f(t, x) + g(t; x) \tag{2.6}$$

where $f(t, x)$ is the function defined by Theorem 2.1 and $g(t; x)$ is the operator defined in section 1. Let us first give the

Definition 2.1. The operator $g(t; x)$ defined for $(t, x) \in (-\infty, \infty) \times AP$, where AP is the Banach space of almost-periodic defined on $(-\infty, \infty)$ is said to be almost-periodic if $\varphi(t) \in AP$ for every function $g(t; \varphi) = (G\varphi)(t) \in AP$.

By using Theorem 2.1 and Banach's fixed-point theorem, we can prove

Theorem 2.2. Suppose that the conditions of Theorem 2.1 are satisfied and that the operator $g(t\,;\,x)$ satisfies the condition

$$\|g(t\,;\,\varphi) - g(t\,;\,\psi)\| \leqslant L\|\varphi - \psi\|_{AP}, \qquad (2.7)$$

for all $(t, \varphi, \psi) \in (-\infty, \infty) \times AP \times AP$.

Then, if $\|A\|L/\alpha < 1$, the system (2.6) has a unique almost-periodic solution.

Proof: For every $\varphi \in AP$, let $T\varphi$ denote the unique almost-periodic solution of the system

$$\dot{x} = f(t, x) + g(t\,;\,\varphi), \qquad (2.8)$$

or, in different notation,

$$\dot{x} = f(t, x) + (G\varphi)(t)\,; \qquad (2.8')$$

(The existence of this solution is asserted by Theorem 2.1). The operator T then has the property that $T(AP) \subset AP$.

Let us show that T is a contraction operator. Let $\varphi, \psi \in AP$ and $x = T\varphi$, $y = T\psi$ denote the solutions of the system (2.6) that correspond to $\varphi(t)$ and $\psi(t)$ respectively.

Consider the function $V(t) = (A[x(t) - y(t)],\ x(t) - y(t))$. We have

$$\frac{1}{2}\dot{V}(t) = (A[\dot{x}(t) - \dot{y}(t)],\ x(t) - y(t)) = (A[f(t, x(t)) - f(t, y(t)) + g(t\,;\,\varphi) -$$
$$- g(t, \psi)],\ x(t) - y(t)) = (A[f(t, x(t)) - f(t, y(t))],\ x(t) - y(t)) +$$
$$+ (A[g(t\,;\,\varphi) - g(t\,;\,\psi)],\ x(t) - y(t)).$$

According to Demidovich's lemma,

$$(A[f(t, x(t)) - f(t, y(t))],\ x(t) - y(t) \leqslant \Lambda(t, x)\|x(t) - y(t)\|^2 \leqslant$$
$$\leqslant -\alpha\|x(t) - y(t)\|^2 \leqslant -\frac{\alpha}{\alpha''}V(t).$$ /383

Similarly, according to Cauchy's inequality,

$$(A[g(t\,;\,\varphi) - g(t\,;\,\psi)],\ x(t) - y(t)) \leqslant \|A\|\,\|g(t\,;\,\varphi) - g(t, \psi)\|\,\|x(t) - y(t)\| \leqslant$$
$$\leqslant \|A\|L\|\varphi - \psi\|_{AP}\ \|x(t) - y(t)\| \leqslant \frac{\|A\|\,L}{\sqrt{\alpha'}}\|\varphi - \psi\|_{AP}\,V^{1/2}(t).$$

Consequently,

$$\frac{1}{2}\dot{V}(t) \leqslant -\frac{\alpha}{\alpha''}V(t) + \frac{\|A\|L}{\sqrt{\alpha'}}\|\varphi - \psi\|_{AP} V^{1/2}(t), \qquad (2.9)$$

or

$$\frac{1}{2}\dot{V}(t) \leqslant -aV(t) + b\|\varphi - \psi\|_{AP} V^{1/2}(t), \qquad (2.9')$$

where

$$a = \frac{\alpha}{\alpha''} > 0, \quad b = \frac{\|A\|L}{\sqrt{\alpha'}} > 0.$$

Using the result on differential inequalities that was used in the proof of Theorem 2.1, we get

$$V(t) \leqslant \sqrt{V(t_0)}\, e^{-a(t-t^0)} + \left[\frac{b\|\varphi - \psi\|_{AP}}{a}\right]^2$$

[Translator's note: There seems to be a mistake in this equation.]
so that

$$V^{1/2}(t) \leqslant \sqrt{V(t_0)}\, e^{-a(t-t_0)} + \frac{b\|\varphi - \psi\|_{AP}}{a}, \qquad (2.10)$$

where

$$V(t_0) = V(x(t_0), y(t_0)).$$

From inequality (2.10), we obtain

$$\|x(t) - y(t)\| \leqslant \sqrt{V(t_0)}\, e^{-a(t-t_0)} + \frac{b}{a\sqrt{\alpha'}}\|\varphi - \psi\|_{AP}, \quad t \geqslant t_0. \qquad (2.11)$$

We shall now show that inequality (2.11) implies

$$\|x(t) - y(t)\| \leqslant \frac{b}{a\sqrt{\alpha'}}\|\varphi - \psi\|_{AP}, \quad t \geqslant t_0. \qquad (2.12)$$

Let us suppose that (2.12) is untrue. Then there exist $\varepsilon < 0$ and $\bar{t} \in (-\infty, \infty)$ /384
such that

$$\|x(\bar{t}) - y(\bar{t})\| \leqslant \frac{b}{a\sqrt{\alpha'}}\|\varphi - \psi\|_{AP} + \frac{\varepsilon}{3}, \quad t \geqslant t_0 + N(\varepsilon).$$

Since $x(t)$ and $y(t)$ are almost-periodic, there exists a $\tau \geq t_0 - t + N(\varepsilon)$ that is an $(\varepsilon/3)$-almost-periodic function of $x(t) - y(t)$ and hence

$$\frac{b}{a\sqrt{\alpha'}}\|\varphi - \psi\|_{AP} + \varepsilon = \|x(\bar{t}) - y(\bar{t})\| \leqslant \|[x(\bar{t}) - y(\bar{t})] - [x(\bar{t} + \tau) -$$

$$- y(\bar{t} + \tau)]\| + \|x(\bar{t} + \tau) - y(\bar{t} + \tau)\| < \frac{2\varepsilon}{3} + \frac{b}{a\sqrt{\alpha'}}\|\varphi - \psi\|_{AP},$$

which implies $\varepsilon < 2\varepsilon/3$. Since this is impossible, inequality (2.12) is proven.

At the same time, inequality (2.12) implies

$$\|T\varphi - T\psi\|_{AP} \leqslant \frac{b}{a\sqrt{\alpha'}}\|\varphi - \psi\|_{AP}, \tag{2.13}$$

that is

$$\rho(T\varphi, T\psi) \leqslant m\rho(\varphi, \psi),$$

where

$$m = \frac{b}{a\sqrt{\alpha'}} = \frac{\|A\|\,L\alpha''}{\alpha(\sqrt{\alpha'})^2} \leqslant \frac{\|A\|\,L}{\alpha} < 1,$$

which proves Theorem 2.2.

Remarks: 1. A similar result can be obtained in the case of periodicity.

2. An interesting particular case is that of the integro-differential equations

$$\dot{x} = f(t, x) + g(t; x)$$

where

$$g(t; x) = \int_{-\infty}^{t} k(t - s)x(s)ds,$$

this operator satisfying Definition 2.2.

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